GRENOBLE 1
UNIVERSITE
JOSEPH FOURIER
SCIENCES. TECHNOLOGIE. SANTÉ

INF121:
Functional Algorithmic and Programming
Lecture 4: Recursion

Academic Year 2011 - 2012

In the previous episodes of INF 121

- Basic Types:

Type	Operations	Constants
Booleans	`not, &&, ||`	`true, false`
Integers	`+,-,*,/,mod, ...`	`...,-1, 0, 1, ...`
floats	`+.,-.,*.,/.`	`0.4, 12.3, 16. , 64.`
char	`lowercase, code, ...`	`'a', 'u', 'A', ...`

- `if` ... `then` ... `else` ... conditional structure
- identifiers (local and global)
- defining and using functions
- Advanced types: synonym, enumerated, product, sum
- Pattern matching on simple and advanced expressions

About recursion

What is recursion/a recursive definition?

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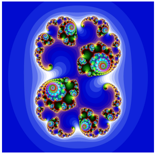
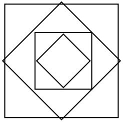
Example (Some recursive objects)

La vache qui rit is a trademark



$u_n, u_{n+1} \ldots$
Fibonacci



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About recursion

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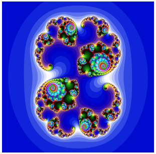

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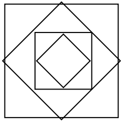

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Recursive functions generalize recursive series

Largely used in Computer Science
$\hookrightarrow$ a computer is a zoo of interacting recursive functions

Outline

- Recursive functions
- Termination
- Recursive types
- Conclusion

Recursive functions in OCaml

An introductory example

Example (Factorial)

$$\begin{cases} 0! = 1 \\ n! = n \times (n-1)!, n \geq 1 \end{cases}$$

$$\begin{aligned} 3! &= 3 \times (3-1)! = 3 \times 2! \\ &= 3 \times 2 \times (2-1)! = 3 \times 2 \times (2-1)! \\ &= 3 \times 2 \times 1 \times (1-1)! = 3 \times 2 \times 1 \times 0! = \ldots = 6 \end{aligned}$$

- This definition is sensible, it allows to obtain a result for all integers: *well-founded*
  (changing the $-$ into $+$ in the 2^{nd} line makes the def not well-founded)

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How can we detect whether a function or a program is well-founded?

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  (changing the − into + in the 2nd line makes the def not well-founded)

How can we detect whether a function or a program is well-founded?

Example (Defining factorial in OCaml)

```
let rec fact (n:int):int =
  if n=0 then 1
  else n ∗ fact(n−1)
```

```
let rec fact (n:int):int =
  match n with
    0 → 1
    | n → n ∗ fact(n−1)
```

Defining a recursive function

Specification: description, signature, examples, and recursive equations

Implementation: defining a recursive function in OCaml

```
let rec fct_name (p1:t1) (p2:t2) ... (pn:tn):t = expr
```

where `expr` generally contains one or more occurrences of `fct_name` s.t.:

- Basis case: no call to the function currently defined
- Recursive calls to the currently defined function (with different parameters)

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Typing works as for non-recursive functions

Remark

- `t1`, .., `tn` can be any type (not necessarily integers) - cf. later
- A recursive function *cannot* be anonymous

□

Defining some recursive functions

Example (Sum of integers from 0 to n)

description + profile + examples

$$\begin{cases} u_0 & = 0 \\ u_n & = n + u_{n-1} \quad \text{when } 0 < n \end{cases}$$

```
let rec sum (n : int) : int =
 match n with
  | 0 → 0
  | n → n + sum (n − 1)
```

Defining some recursive functions

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```
let rec sum (n : int) : int =
 match n with
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```

Example (Quotient of the Euclidian division)

description + profile + examples

$$a/b = \begin{cases} 0 & \text{when } a < b \\ 1 + (a - b)/b & \text{when } b \leq a \end{cases}$$

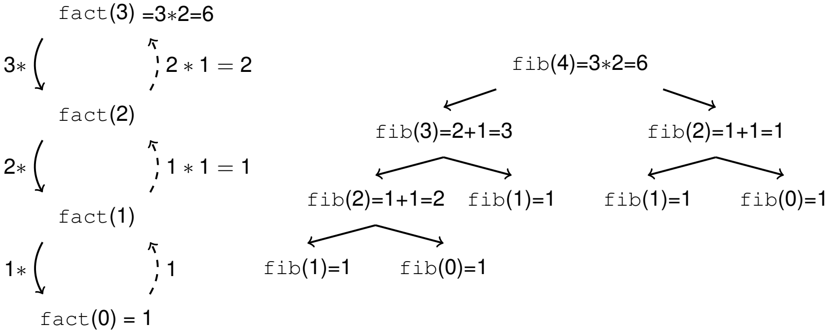
```
let rec div (a : int) (b: int) : int =
 if a < b then 0
  else 1 + div (a − b) (b)
```

DEMO: some other recursive functions

Calling a recursive function

"Unfolding the function body" – rewriting

Example (Factorial and Fibonacci's call trees)



`fact`(3) =3∗2=6

3∗ … $2 * 1 = 2$

`fact`(2)

2∗ … $1 * 1 = 1$

`fact`(1)

1∗ … 1

`fact`(0) = 1

`fib`(4)=3∗2=6

`fib`(3)=2+1=3

`fib`(2)=1+1=1

`fib`(2)=1+1=2

`fib`(1)=1

`fib`(1)=1

`fib`(0)=1

`fib`(1)=1

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- $\longrightarrow$: rewriting generated calls and suspending operations
- $\dashrightarrow$: evaluation (in the reverse order) of suspended operations

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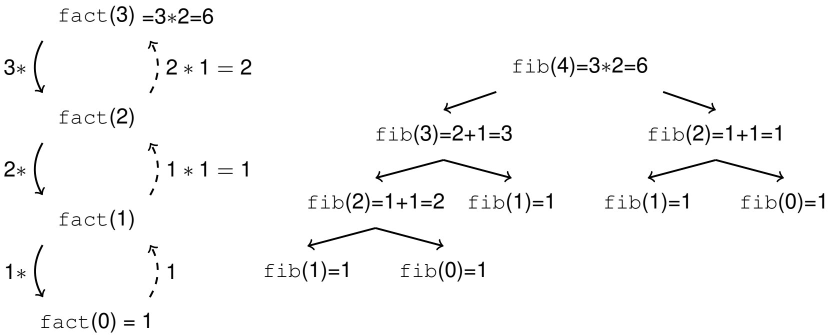

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`fact`(1)

1∗ 1

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`fib`(3)=2+1=3

`fib`(2)=1+1=1

`fib`(2)=1+1=2

`fib`(1)=1

`fib`(1)=1

`fib`(0)=1

`fib`(1)=1

`fib`(0)=1

- $\longrightarrow$: rewriting generated calls and suspending operations
- $\dashrightarrow$: evaluation (in the reverse order) of suspended operations

In OCaml: directive #`trace`

DEMO: Tracing a function

Let's practice

Exercise: remainder of the Euclidean division

Define a function which computes the remainder of the Euclidean division

Exercise: The Fibonacci series

Implement a function which returns the n^{th} Fibonacci number where n is given as a parameter. Formally the Fibonacci series is defined as follows:

$$fib_n = \begin{cases} 1 & \text{when } n = 0 \text{ or } n = 1 \\ fib_{n-1} + fib_{n-2} & \text{when } n > 1 \end{cases}$$

Let’s practice

Exercise: the power function (two ways)

$$\begin{cases} x^0 & = 1 \\ x^n & = x * x^{n-1} \quad \text{when } 0 < n \end{cases} \qquad \begin{cases} x^0 & = 1 \\ x^n & = (x * x)^{n/2} \quad \text{when } n \text{ is even} \\ x^n & = x * (x * x)^{\frac{n-1}{2}} \quad \text{when } n \text{ is odd} \end{cases}$$

- Define function `power`: `int` → `int` → `int` twice following the two equivalent mathematical definitions
- What is the difference between those two versions?

Let's practice

Exercise: the power function (two ways)

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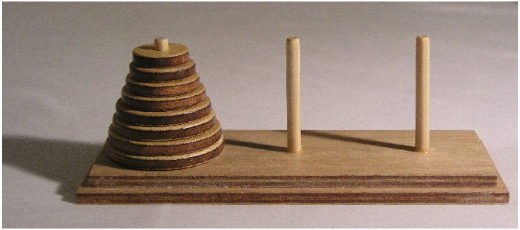
- Define function `power`: `int` → `int` → `int` twice following the two equivalent mathematical definitions
- What is the difference between those two versions?

```
let rec pow (x:float) (n:int):int =
  if (n=0) then 1
  else x * (pow x n−1)
```

```
let rec pow (x:float) (n:int):int =
 if (n=0) then 1
 else (
  if (n mod 2=0) then (pow (x*x) (n/2))
  else pow (x*x) ((n−1)/2)
 )
```

The Hanoi towers

A word about Divide and Conquer



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Mutually recursive functions

On an example

So far “direct” recursion: a function `fct` contains calls to itself

What about a function `f` which calls `g` which calls `f`
↪ mutually recursive functions

Mutually recursive functions

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Example (Is a number odd or even)

How to determine whether an integer is odd or even without using
/, $*$, `mod`,and, more specifically using $-$ and =?

Mutually recursive functions

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Example (Is a number odd or even)

How to determine whether an integer is odd or even without using /, $*$, `mod`,and, more specifically using $-$ and =?

- $n \in \mathbb{N}$ is odd if $n - 1$ is even
- $n \in \mathbb{N}$ is even if $n - 1$ is odd
- 0 is even
- 0 is not odd

Mutually recursive functions

On an example

So far “direct” recursion: a function `fct` contains calls to itself

What about a function `f` which calls `g` which calls `f`
$\hookrightarrow$ mutually recursive functions

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How to determine whether an integer is odd or even without using /, $*$, `mod`,and, more specifically using $-$ and =?

- $n \in \mathbb{N}$ is odd if $n-1$ is even
- $n \in \mathbb{N}$ is even if $n-1$ is odd
- 0 is even
- 0 is not odd

```
let rec even (n:int):bool = if n=0 then true else odd (n−1)
  and odd (m:int):bool = if m=0 then false else even (m−1)
```

DEMO: even and odd, mutually recursive

Mutually recursive functions

Generalization

Mutually recursive functions

```
let rec fct1 [parameters+return type] = expr_1
  and fct2 [parameters+return type] = expr_2
     ....
  and fctn [parameters+return type] = expr_n
```

`expr_1`, `expr_2`, ..., `expr_n` may have calls to `fct1`, `fct2`, ..., `fctn`

Outline

Recursive functions

Termination

Recursive types

Conclusion

Termination

Do you think this function terminates (the McCarthy function)?

$$mac(n) = \begin{cases} n - 10 & \text{when } n > 100 \\ mac(mac(n + 11)) & \text{when } n \leq 100 \end{cases}$$

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What about these ones?

The power function

$$\begin{cases} x^0 & = 1 \\ x^n & = x * x^{n-1} \quad \text{when } 0 < n \end{cases}$$

The factorial function

$$\begin{cases} fact(0) & 1 \\ fact(1) & = 1 \\ fact(n) & = \frac{fact(n+1)}{n+1} \end{cases}$$

Termination

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The factorial function

$$\begin{cases} fact(0) & 1 \\ fact(1) & = 1 \\ fact(n) & = \frac{fact(n+1)}{n+1} \end{cases}$$

We are only interested in terminating functions...

Can we have an intuitive characterization of termination w.r.t. the calling tree?

How can we prove that a recursive function terminate?

Using a Measurement

Theorem

Every series of positive numbers which is <u>strictly</u> decreasing is converging

How can we prove that a recursive function terminate?

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Theorem

Every series of positive numbers which is strictly decreasing is converging

General methodology to show a function is terminating

From the def. of the function and its parameters, derive a measurement s.t.:

- it is positive
- the measurement *strictly decreases* between two recursive calls

each recursive call “brings us closer to the base case”

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From the def. of the function and its parameters, derive a measurement s.t.:

- it is positive
- the measurement *strictly decreases* between two recursive calls

each recursive call "brings us closer to the base case"

Example (Termination of the function `sum`)

```
let rec sum (n : int) : int =
  match n with
   | 0 → 0
   | n → n + sum (n − 1)
```

Measurement:

- Let's define $\mathcal{M}(n) = n$
- $\mathcal{M}(n) \in \mathbb{N}$ (according to the spec)
- $\mathcal{M}(n) > \mathcal{M}(n-1)$ since $n > n-1$

Termination of some functions

Exercise: finding measurements

Revisit the functions `factorial`, `power`, `quotient`, `remainder` and find the measurement proving that your function terminates

Termination of some functions

factorial and power

Termination of `fact`:

```
let rec fact (n:int):int =
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Termination of some functions

factorial and power

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Termination of some functions

factorial and power

Termination of `fact`:

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let rec fact (n:int):int =
  match n with
    0 → 1
    | n → n * fact(n−1)
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- Let's define $\mathcal{M}(\mathit{fact}\ n) = n$
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Termination of `power`:

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let rec power (a:float) (n:int):float =
  if (n=0) then 1.
 else (if n>0 then a *. power  a (n−1)
    else 1. /. (power a (n−1))
  )
```

Termination of some functions

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 else (if n>0 then a ∗. power  a (n−1)
    else 1. /. (power a (n−1))
  )
```

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- $\mathcal{M}(\mathit{power}\ a\ n) \in \mathbb{N}$ (according to the spec)
- $\mathcal{M}(\mathit{power}\ a\ n) > \mathcal{M}(\mathit{power}\ a\ (n-1))$

Termination of some functions

```
let rec quotient (a:int) (b:int):int =
 if (a<b) then 0
 else 1 + quotient (a−b) b
```

```
let rec remainder (a:int) (b:int):int =
  if (a<b) then a
  else remainder (a−b) b
```

Termination of some functions

```
let rec quotient (a:int) (b:int):int =
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let rec remainder (a:int) (b:int):int =
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```

Termination of `quotient` and `remainder`:

- Let's define $\mathcal{M}(X\,a\,b) = a$
- $\mathcal{M}(X\,a\,b) \in \mathbb{N}$ (according to the spec)
- $\mathcal{M}(X\,a\,b) > \mathcal{M}(X\,(a-b)\,b)$ since $b > 0$

where $X \in \{\text{quotient}, \text{remainder}\}$

Outline

- Recursive functions
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- Recursive types
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Recursive types

Recursive functions are functions that appear in their own definition

Recursive types are types that appear in their own definition

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General syntax: `type new_type` = ... `new_type`...

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Recursive types should be well-founded

They make sense only for Union type with a non recursive constructor (constant or not)

DEMO: (not) Well-founded types

DEMO: Metaphor of building a wall

Recursive types

Recursive functions are functions that appear in their own definition

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General syntax: `type new_type` = ... `new_type`...

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DEMO: (not) Well-founded types

DEMO: Metaphor of building a wall

Definition of a recursive function on a recursive type should follow the recursive type

A recursive type: Peano natural numbers

The mathematical and OCaml perspectives

Peano natural numbers $\mathrm{NatPeano}$: an alternative definition of $\mathbb{N}$

Recursive definition of $\mathrm{NatPeano}$:

- a basis natural Zero
- a *constructor*: Suc: returns the successor of a $\mathrm{NatPeano}$ number
- Zero is the successor of no $\mathrm{NatPeano}$ number
- two $\mathrm{NatPeano}$ numbers having the same successor are equal

$\hookrightarrow$ $\mathbb{N}$ can be defined as the set containing Zero and the successor of any element it contains

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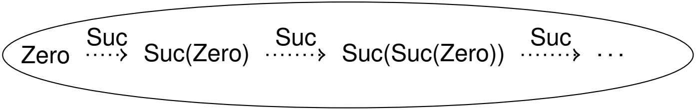

$$\begin{tikzcd} \text{Zero} \arrow[r, dotted, "\text{Suc}"] & \text{Suc(Zero)} \arrow[r, dotted, "\text{Suc}"] & \text{Suc(Suc(Zero))} \arrow[r, dotted, "\text{Suc}"] & \cdots \end{tikzcd}$$

Defining $\mathrm{NatPeano}$ in OCaml:

```
type natPeano = Zero | Suc of natPeano
```

$\hookrightarrow$ `natPeano` is a **recursive sum type**

Peano natural numbers

Conversion to and from integers

Example (Converting a Peano natural number into an integer)

- Description: `natPeano2int` translates a Peano number into its usual counterpart in the set of integers
- Profile/Signature: `natPeano2int`: `natPeano` → `int`
- Ex.: `natPeano2int Zero` = 0, `natPeano2int Suc(Suc(Suc Zero))`=3

```
let rec natPeano2int (n:natPeano):int =
 match n with
    Zero → 0
   | Suc (nprime) → 1+ natPeano2int nprime
```

Peano natural numbers

Conversion to and from integers

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- Profile/Signature: `natPeano2int: natPeano → int`
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```
let rec natPeano2int (n:natPeano):int =
 match n with
    Zero → 0
   | Suc (nprime) → 1+ natPeano2int nprime
```

Example (Converting an integer into a Peano number)

Same as above but in the converse sense:

```
let rec int2natPeano (n:int):natPeano=
  match n with
   0 → Zero
   | nprime → Suc (int2natPeano (n−1))
```

DEMO: Functions natPeano2int and int2natPeano

Peano natural numbers

Some functions: sum, product

Exercise: sum of two Peano numbers

- Define the function that *sums two Peano numbers* without using the conversion from/to int
- Prove that your function terminates

Exercise: product of two Peano numbers

- Define the function that *multiplies two Peano numbers*
- Prove that your function terminates

Exercise: factorial of a Peano number

- Define the function that computes the *factorial* of a Peano number
- Prove that your function terminates

A recursive type: polynomials of 1 variable

A polynomial of one variable (a sum of monomials):

$$\alpha_n X^n + \alpha_{n-1} X^{n-1} + \ldots + \alpha_1 X^1 + \alpha_0$$

A recursive type: polynomials of 1 variable

A polynomial of one variable (a sum of monomials):

$$\alpha_n X^n + \alpha_{n-1} X^{n-1} + \ldots + \alpha_1 X^1 + \alpha_0$$

Let's see it as a recursive object: a polynomial is either a monomial or the sum of monomial and another polynomial

Model 1:

```
type coef = int
type degree = int
type monomial = coef * degree
type polynomial = Mn of monomial
          | Plus of monomial * polynomial
```

DEMO: Model 1 of Polynomials + its disadvantages

A recursive type: polynomials of 1 variable - ctd

Model 2:

- with canonical representation
- no monomial with null coefficient

```
type polynomial = Zero | Plus of monomial * polynomial

let well_formed (p:polynomial):bool = ...
(* checks order of coef + no null coeff *)
```

A recursive type: polynomials of 1 variable - ctd

Model 2:

- with canonical representation
- no monomial with null coefficient

```
type polynomial = Zero | Plus of monomial * polynomial
```

```
let well_formed (p:polynomial):bool = ...
(* checks order of coef + no null coeff *)
```

Exercise: Some functions around polynomials

- Define a function that checks whether a polynomial is well-formed, by:
  - checking that there is no null coefficient
  - degrees are given in decreasing order
- Degree max: Propose a new implementation of the function degree max supposing that a polynomial is well-formed
- Addition of two polynomials:
  - Define a function that performs the addition between a polynomial and a monomial
  - Define a function that performs the addition between two polynomials

Conclusion

Recusion: a fundamental notion

There are two forms of recursion in computer science:

- recursive functions
    - recursive equations
    - termination
    - definition = spec (description, profile, recursive equations, examples) + implem + terminations
    - pitfalls
- Recursive types/values/objects
    - definition
- Recursive functions on recursive types